Reply to the 'Comment on 'On the uncertainty relations and squeezed states for the quantum mechanics on a circle''

**REPLY**

# Reply to the 'Comment on 'On the uncertainty relations and squeezed states for the quantum mechanics on a circle''

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In the preceding comment [1] Trifonov disputes our uncertainty relations for a quantum particle on a circle recently proposed in [2] such that

$$\Delta^2(\hat{\varphi}) + \Delta^2(\hat{J}) \geqslant 1 \tag{1}$$

where $\Delta^2(\hat{\varphi})$ and $\Delta^2(\hat{J})$ are measures of the uncertainty of the position and angular momentum, respectively. He states that (i) the quantity $\Delta^2(\hat{\varphi})$ introduced in [2] representing the uncertainty of the angle is not a proper measure of the position uncertainty and therefore the proposed inequality (1) can hardly be qualified as a relevant uncertainty relation on a circle; and that (ii) the most suitable uncertainty relations on a circle are those based on the Gram–Robertson matrix [3]. We disagree with both points.

(i) We recall that Trifonov [1] provides an example of the state which can be regarded as a counterpart of the Schrödinger cat state in the case of the circular motion, such that the corresponding wave packet seems to be more poorly localized than that referring to the coherent state for a quantum particle on a circle, despite the fact that the uncertainty $\Delta^2(\hat{\varphi})$ in the Schrödinger cat state is less than that in the coherent state. In our opinion the discussion of the uncertainty relations cannot be confined, as done by Trifonov [1], to the localization in the configuration space but it must take into consideration the localization in the *phase space*. Reasoning analogously as Trifonov, one could provide the following 'proof' of the irrelevance of the uncertainty relations for the sum of variances of the position and momentum of a particle on a real line implied by the standard Heisenberg uncertainty relations, of the form

$$\Delta^2 \hat{x} + \Delta^2 \hat{p} \geqslant 1 \tag{2}$$

where we set $\hbar = 1$.

Consider the wavefunctions such that [4]

$$\psi(x) = \begin{cases} 1/\sqrt{L} & \text{for} \quad -L/2 < x < L/2 \\ 0 & \text{for} \quad -L/2 > x > L/2 \end{cases} \tag{3}$$

$$\phi(x) = \begin{cases} \sqrt{2/L} & \text{for} & -L/2 < x < -L/4 \\ 0 & \text{for} & -L/4 < x < L/4 \\ \sqrt{2/L} & \text{for} & L/4 < x < L/2 \\ 0 & \text{for} & -L/2 > x > L/2 \end{cases} \tag{4}$$

where $L > 0$. As one can see the state $|\psi\rangle$ is much more poorly localized on the interval $|x| < L/2$, than the state $|\phi\rangle$. In fact, we know that in the state $|\phi\rangle$ the particle is not in the region $|x| < L/4$. However, when we calculate the variances we get

$$\Delta_\psi^2 \hat{x} = \frac{L^2}{12} \qquad \Delta_\phi^2 \hat{x} = \frac{7}{4} \frac{L^2}{12}. \tag{5}$$

Thus it turns out that that the variance in the state $|\psi\rangle$ is considerably less than that in the state $|\phi\rangle$. Therefore, concluding the 'proof'—the variance is not a proper measure of the position uncertainty and the Heisenberg uncertainty relations could hardly be qualified as a relevant uncertainty relations on a line.

Finally, we would like to stress that the motivation for the usage in [2] of the denomination 'squeezed states' was only the formal similarity of generation of these states and the standard squeezed states. In particular, neither any squeezing property was discussed nor any definition provided in [2] like 'The quantity $\tilde{\Delta}^2(\hat\phi)$ is called squeezed if it is less than $1/2$' as erroneously indicated in [1]. Moreover, the problems were reported in [2] with the physical interpretation of the parameter $s$ labelling the squeezed states for the quantum mechanics on a circle.

(ii) The uncertainty relations on a circle proposed by Trifonov [1] utilize (generalized) variances of the angle. We share the opinion of Białynicki-Birula et al [5] that: 'Second moment or variance .... This is a naive extension of the mathematical formulation of uncertainty which is used for Heisenberg's position–momentum uncertainty relation. The main drawback of this measure of uncertainty is that we evaluate the averages of non-periodic function, such as $\varphi$ or $\varphi^2$, with a periodic distribution function. Consequently this measure can assume completely arbitrary values depending on the origin of the phase integration, that is on the coordinatization of the unit circle'. In fact, since there is no distinguished point on a circle, therefore it is clear that the uncertainty of the position of a quantum particle should depend solely on its state and not the choice of the particular point on a circle. Evidently, this is not the case when we apply the standard variance. Namely, we find

$$\Delta_\lambda^2 \hat\varphi - \Delta_0^2 \hat\varphi = 2 \int_0^\lambda (\varphi + \pi - \langle\varphi\rangle_0) |f(\varphi)|^2 \, \mathrm{d}\varphi - \left( \int_0^\lambda |f(\varphi)|^2 \, \mathrm{d}\varphi \right)^2 \tag{6}$$

where

$$\Delta_\lambda^2 \hat\varphi = \langle \hat\varphi^2 \rangle_\lambda - \langle \hat\varphi \rangle_\lambda^2 = \frac{1}{2\pi} \int_\lambda^{2\pi+\lambda} \varphi^2 |f(\varphi)|^2 \, \mathrm{d}\varphi - \left( \frac{1}{2\pi} \int_\lambda^{2\pi+\lambda} \varphi |f(\varphi)|^2 \, \mathrm{d}\varphi \right)^2 \tag{7}$$

and the normalized wave packet $f(\varphi)$ is a $2\pi$-periodic function, i.e. $f(\varphi + 2\pi) = f(\varphi)$. For an easy illustration of the dependence of the variance $\Delta_\lambda^2 \hat\varphi$ on the origin of integration $\lambda$ we now discuss the case of the normalized wave packet of the form

$$f(\varphi) = \sqrt{\frac{2\pi}{\epsilon}} \chi_{[0,\epsilon]}(\varphi) \tag{8}$$

where $\chi_{[0,\epsilon]}(\varphi)$ is the characteristic function of the interval $[0, \epsilon]$ and $\epsilon \in (0, 2\pi)$. Of course, the wave packet (8) can be made $2\pi$-periodic by taking the interval $[0, \epsilon]$ modulo $2\pi$. Now the straightforward calculation shows that the difference of variances (6) for the wave packet (8) is

$$\Delta_\lambda^2 \hat\varphi - \Delta_0^2 \hat\varphi = \begin{cases} 0 & \text{for} \quad \epsilon \leqslant \lambda \\ \frac{2\pi}{\epsilon} \lambda \left[ \left(1 - \frac{2\pi}{\epsilon}\right) \lambda + 2 \left(\pi - \frac{\epsilon}{2}\right) \right] & \text{for} \quad \epsilon > \lambda. \end{cases} \tag{9}$$

Thus, as expected, the difference of variances (6) depends in general on the origin of integration $\lambda$. We would like to point out that in a sense Trifonov seems to recognize the discussed flaw of the standard variance since he suggests in [1] that 'the mean values $\langle \varphi \rangle$, $\langle \varphi^2 \rangle$ should be calculated by integration from $\varphi_0 - \pi$ to $\varphi_0 + \pi$, where $\varphi_0$ is the centre of the wave packet (i.e. $\varphi_0$ is the most probable value of $\varphi$)'. In our opinion such a solution of the problem which introduces the *definition* of average values depending on the particular state of the system can hardly be called satisfactory. Another evidence that the variance utilized by Trifonov [1] can hardly be qualified as a relevant uncertainty of the position on a circle is the ill behaviour of the expectation value $\langle \hat{\varphi}(t) \rangle$ in the case of the free evolution of the coherent states. Namely, it turns out that $\langle \hat{\varphi}(t) \rangle$ takes the values only from the subset of the circle $[0, 2\pi)$ [6].

We would like to stress that our measure of the uncertainty of the position of a quantum particle on a circle given by [2]

$$\Delta^2(\hat{\varphi}) = -\tfrac{1}{4} \ln|\langle U^2 \rangle|^2 \tag{10}$$

where $U = \exp(i\hat{\varphi})$, has correct behaviour and does not depend on the origin of the integration. Indeed, we have

$$\Delta_\lambda^2(\hat{\varphi}) = \Delta_0^2(\hat{\varphi}) \tag{11}$$

where

$$\Delta_\lambda^2(\hat{\varphi}) = -\tfrac{1}{4} \ln|\langle U^2 \rangle_\lambda|^2 \tag{12}$$

and

$$\langle U^2 \rangle_\lambda = \frac{1}{2\pi} \int_\lambda^{2\pi+\lambda} e^{2i\varphi} |f(\varphi)|^2 \, d\varphi \tag{13}$$

following immediately from

$$\langle U^2 \rangle_\lambda = \langle U^2 \rangle_0. \tag{14}$$

We remark that an interesting observation of Trifonov [1] is that the uncertainty relations are minimized by the Schrödinger-cat-like states mentioned earlier. Therefore, in opposition to the standard coherent states, the coherent states for the quantum mechanics on a circle are not uniquely determined, up to a unitary transformation, by the requirement of the saturation of the uncertainty relations (1). Nevertheless, the topology of the circle is completely different from the topology of the real line and it seems plausible that the coherent states for a quantum particle on a circle may have some properties different from those of the standard coherent states referring to the case of the real line. We also point out that Trifonov has not provided in [1] any example of states violating the inequality (1).

Finally, we would like to comment on the note added to proof [1], that '...coherent states have been introduced (in more general notations) by S de Bievre and J Gonzales in 1993 [2]'. First of all, we introduced in [7] the coherent states for a quantum particle on a circle as a solution of some eigenvalue equation independently of the treatment of Gonzales *et al* [8] who applied the Weil–Brezin–Zak transform. We stress that our approach based on a polar decomposition of an operator defining via the eigenvalue equation the coherent states enabled us to construct the coherent states for the quantum mechanics on a sphere [9]. We remark that both coherent states for a quantum particle on a circle and on a sphere are concrete realizations of the general mathematical scheme of construction of the Bargmann spaces introduced in [10, 11] (see also recent work [12]). The 'more general notations' mentioned by Trifonov are connected with the fact that the coherent states utilized by Gonzales *et al* [8] are labelled by some parameter which can be avoided by demanding the time-reversal invariance [7] which leads precisely to the coherent states introduced in [7].

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